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Mean Approximation by Transformed and Constrained Rational Functions

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The problem of existence of best approximations by transformed and constrained rational functions with respect to a generalized integral norm is studied.

Let X be a compact topological space which is also a measure space and let $\int$ denote the integral over X. Let τ be a continuous mapping of the real line into the nonnegative real line. For a real (finite) measurable g, defined on X, set

$$\| g \| = \int \tau(g).$$

Let $\{\phi_1, \ldots, \phi_n\}$, $\{\psi_1, \ldots, \psi_m\}$ be linearly independent subsets of $C(X)$. Define

$$R(A, x) = P(A, x)/Q(A, x) = \sum_{k=1}^{n} a_k \phi_k(x) \Big/ \sum_{k=1}^{m} a_{n+k} \psi_k(x).$$

Let σ be a continuous mapping of the real line into itself. Define

$$F(A, x) = \sigma(R(A, x)).$$

Let P be a subset of $n + m$ space. The approximation problem is: given f, finite, measurable, to find an $A^* \in P$ for which $\| f - F(A, \cdot)\|$ attains its infimum

$$\rho(f) = \inf\{\| f - F(A, \cdot)\| : A \in P\}.$$

Such a parameter A^* is called best and $F(A^*, \cdot)$ is called a best approximation of f.

The study of linear approximation by τ-"norms" was begun by Walsh and Motzkin [5]. The case where X is an interval, $\sigma(x) = x$, and the only constraint on the parameters A is that $Q(A, \cdot) \neq 0$ is considered in [2]. Cases in which a weight function is used are handled by incorporating the weight function into the measure or integral.

Q WITH THE ZERO MEASURE PROPERTY

In case $Q(A, x) \neq 0$, $F(A, x)$ is well defined. We need a convention for cases in which $Q(A, x)$ has zeros x. We use a hypothesis of Boehm [1] as adapted in [2].

DEFINITION. Q has the *zero measure property* if $Q(A, \cdot) \not\equiv 0$ implies that the set of zeros of $Q(A, \cdot)$ is of zero measure.

EXAMPLE. Let $X = [0, 1] \times [0, 1]$ and $Q(A, (x, y)) = a_{n+1} + a_{n+2}x + a_{n+3}y$, then if $Q(A, \cdot) \not\equiv 0$, the zeros of $Q(A, \cdot)$ form at most a line segment in X.

If this condition holds, $F(A, \cdot)$ may need an extra definition on a set of measure zero, if $Q(A, \cdot) \not\equiv 0$. But the values of $F(A, \cdot)$ on a set of measure zero have no effect on the value of $\int \tau(f - F(A, \cdot))$, so it does not matter how we define $F(A, x)$ for the zeros x of $Q(A, x)$.

Since $R(\alpha A, x) = R(A, x)$ for all $\alpha > 0$, any rational which does not have its denominator vanishing identically can be normalized so that

$$\sum_{k=1}^{m} | a_{n+k} | = 1. \tag{1}$$

Define P_0 to be the set of parameters A satisfying (1) and $Q(A, \cdot) \geqslant 0$.

LEMMA 1. *Let Q have the zero measure property and there exist B such that $Q(B, \cdot) > 0$. Let $Q(A, \cdot) \geqslant 0$, $Q(A, \cdot) \not\equiv 0$, then $R(A, \cdot)$ is measurable.*

Proof. If $Q(A, \cdot) > 0$, $R(A, \cdot)$ is continuous and, therefore, measurable. If $Q(A, \cdot) \geqslant 0$, $Q(A, \cdot) \not\equiv 0$, define

$$R(A^k, x) = R\left(\frac{k-1}{k} A + \frac{1}{k} B, x\right)$$

then $Q(A^k, \cdot) > 0$, hence $R(A^k, \cdot) \in C(X)$, $R(A^k, \cdot)$ measurable, and $R(A^k, x)$ converges to $R(A, x)$ if $Q(A, x) = 0$, hence $R(A, \cdot)$ is measurable [3, p. 43].

COROLLARY. *Under the same hypotheses, $F(A, \cdot)$ is measurable.*

The analog of Lemma 2 of [2] follows.

LEMMA 2. *Let $| \sigma(t)| \to \infty$ as $| t | \to \infty$. If $\{\| A^k \|\} \to \infty$ then there is a closed neighborhood N in X such that*

$$\inf\{| f(x) - \sigma(R(A^k, x))| : x \in N\} \to \infty \quad \textit{as} \quad k \to \infty.$$

THEOREM 1. *Let Q have the zero measure property and there exist B with $Q(B, \cdot) > 0$. Let $\tau(t) \to \infty$ as $|t| \to \infty$ and $|\sigma(t)| \to \infty$ as $|t| \to \infty$. Let neighborhoods be of positive measure. Let P be a nonempty closed subset of P_0. There exists a best approximation to each bounded measurable function.*

Proof. Let $\|f - F(A^k, \cdot)\|$ be a decreasing sequence with limit $\rho(f)$. By Lemma 2 it can be easily seen that $\{\| A^k \|\}$ must be a bounded sequence. Thus, $\{A^k\}$ has an accumulation point A^0, assume without loss of generality that $\{A^k\} \to A^0$. As P is closed, $A^0 \in P$ and

$$\sum_{k=1}^{n} | a_{n+k}^0 | = 1.$$

It follows that the set of zeros of $Q(A^0, \cdot)$ is of measure zero. If $Q(A^0, x) \neq 0$, $R(A^k, x) \to R(A^0, x)$ and $|f(x) - F(A^k, x)| \to |f(x) - F(A^0, x)|$. By Fatou's theorem [3, p. 59],

$$\|f - F(A^0, \cdot)\| \leqslant \limsup_{k\to\infty} \|f - F(A^k, \cdot)\| = \rho(f).$$

PARAMETER SPACES

We now consider some subsets of P_0 under the assumption that B exists so that $Q(B, \cdot) > 0$.

(1) P_0 is a closed nonempty set.

(2) Let $\{x_1, ..., x_p\}$ be a finite subset of X and $\{y_1, ..., y_p\}$ be real numbers. Define

$$P_1 = \{A: F(A, x_i) = y_i, i = 1, ..., p\}.$$

When the convention of Boehm [1] is used to assign values to rational functions, P_1 need not be closed and best approximations need not exist.

EXAMPLE. Let $\sigma(x) = x$, $R(A, x) = a_1/(a_2 + a_3 x)$. Let

$$A^k = (1/k, 1/k, (k-1)/k)$$

then $R(A^k, 0) = 1$. We have $\{A^k\} \to (0, 0, 1) = A^0$ and since $R(A^0, x) = 0$ for $x \neq 0$, $R(A^0, 0) = 0$ by Boehm's convention. Let us approximate f:

$$f(x) = 1, \quad x = 0;$$
$$= 0, \quad x > 0,$$

on [0, 1] under the constraint $R(A, 0) = 1$. As $\|f - R(A^k, \cdot)\| \to 0$, a best A would satisfy $\|f - R(A, \cdot)\| = 0$. The only rational $R(A, \cdot)$ for which this is true is the zero function, which does not satisfy the constraint.

Goldstein has used a convention [4, pp. 84–89] in which $R(A, x)$ is assigned *any* desired value when $P(A, x) = Q(A, x) = 0$. With this convention P_1 can be made closed. Let $\{A^k\}$ satisfy the constraints

$$F(A^k, x_i) = y_i, \qquad i = 1,\dots, p, \tag{2}$$

and $\{A^k\} \to A$. If $Q(A, x_i) \neq 0$, $F(A^k, x_i) \to F(A, x_i)$. If $Q(A, x_i) = 0$, $P(A, x_i) \neq 0$, then $|F(A^k, x_i)| \to \infty$. If $P(A, x_i) = Q(A, x_i) = 0$ we assign to $F(A, x_i)$ the value y_i. It follows that P_1 is closed. As denominators are not a problem in linear approximation ($m = 1$), P_1 is closed in transformed linear approximation.

(3) Let u, v be functions mapping X into the extended real line, $u \leqslant v$, and

$$P_2 = \{A: u \leqslant F(A, \cdot) \leqslant v\}.$$

This choice of parameters is associated with the problem of constrained approximation. Special cases of interest are those of one-sided approximation in which $u = -\infty$, $v = f$ or $u = f$, $v = +\infty$. In dealing with P_2 we use also the convention of Boehm [1].

LEMMA 3. *Let Q have the nonzero dense property and Boehm's convention be used. Let u be lower semicontinuous into the extended real line and v be upper semicontinuous into the extended real line, then $P_2 \cap P_0$ is closed.*

Proof. Let $\{A^k\}$ be a sequence in $P_2 \cap P_0$ and $\{A^k\} \to A$. Let $Q(A, x) \neq 0$, then $\{R(A^k, x)\} \to R(A, x)$, hence $\{F(A^k, x)\} \to F(A, x)$. We, therefore, have $u(x) \leqslant F(A, x) \leqslant v(x)$ for such x. Let $Q(A, x) = 0$. There exists a sequence $\{x_k\} \to x$ such that $Q(A, x_k) \neq 0$ and

$$\limsup\{R(A, y): y \to x, Q(A, y) \neq 0\} = \lim_{k\to\infty} R(A, x_k),$$

hence

$$F(A, x) = \sigma(R(A, x)) = \sigma(\lim_{x_k \to x} R(A, x_k)) = \lim_{x_k \to x} \sigma(R(A, x_k)).$$

But $\sigma(R(A, x_k) \geqslant u(x_k)$ so by lower semicontinuity of u, $\sigma(R(A, x) \geqslant u(x)$ Similarly $\sigma(R(A, x)) \leqslant v(x)$.

(4) Let $J = \{j_1, \dots, j_p\}$ be a subset of $\{1, 2,\dots, n + m\}$, and let $\{s_1, \dots, s_p\}$ be a set of signs ($+1$ or -1). Let P_3 be the set of coefficient vectors A such that

$$\operatorname{sgn}(a_k) = s_k \text{ or } 0, \qquad k \in J.$$

P_3 is closed, hence $P_0 \cap P_3$ is closed. A special case is where all coefficients of A are to be nonnegative [6].

(5) Let X be a compact subset of the real line and Y be a closed subset of X. Let P_4 be the set of coefficient vectors A such that $R(A, \cdot)$ is monotonic increasing on Y. If Boehm's convention [1] can be used on Y (which implies that Y has no isolated points) then $P_4 \cap P_0$ is closed.

Suppose not then there exists a sequence $\{A^k\} \subset P_4 \cap P_0$ and $A \notin P_4$ such that $\{A^k\} \to A$. Hence there are points $x, y \in Y$, $x < y$ and $\epsilon > 0$ such that $R(A, x) - R(A, y) > \epsilon$. By Boehm's convention there are x', $y' \in Y$, $x' < y'$ such that $Q(A, x') > 0$, $Q(A, y') > 0$, and $R(A, x') - R(A, y') > \epsilon/2$. For all k sufficiently large we have $R(A^k, x') - R(A^k, y') > \epsilon/4$, contradicting monotonicity of $R(A^k, \cdot)$ on Y.

We may want $F(A, \cdot)$ to be monotonic. If σ is monotonic we need merely make $R(A, \cdot)$ monotonic.

Admissible Approximation

A transformed rational function is called *admissible* if it can be expressed as $\sigma(R(A, \cdot))$, $Q(A, \cdot) > 0$. In some cases we can show that a best approximation exists which is admissible, and hence the problem of approximation by admissible transformed rational functions has a solution.

Definition. (R, P) has the *admissible property* if for given $A \in P$, $\int \tau(f - F(A, \cdot)) < \infty$ implies that there is $B \in P$, $Q(B, \cdot) > 0$ with $R(A, \cdot) - R(B, \cdot) = 0$ almost everywhere.

Corollary. *Let the hypotheses of the theorem hold and (R, P) have the admissible property. There exists a best admissible approximation to all measurable bounded functions.*

Proof. By the theorem there exists a best approximation $F(A, \cdot)$, $A \in P$. If $\int \tau(f - F(A, \cdot)) < \infty$ there is $B \in P$, $Q(B, \cdot) > 0$ such that $F(B, \cdot) - F(A, \cdot) = 0$ almost everywhere, and hence $\int \tau(f - F(A, \cdot)) = \int \tau(f - F(B, \cdot))$. We apply the corollary to the most common case of interest, which covers all L_p norms, $1 \leqslant p < \infty$, on an interval $X = [a, b]$.

Theorem. *Let there exist α, K such that $\tau(t) \geqslant \alpha \mid t \mid$ for all $\mid t \mid \geqslant K$. Let there exist β, M such that $\mid \sigma(t) \mid \geqslant \beta \mid t \mid$ for $\mid t \mid \geqslant M$. Let f be a bounded measurable function on $[a, b]$. Let*

$$R(A, x) = P(A, x)/Q(A, x) = \sum_{k=1}^{n} a_k x^{k-1} \Big/ \sum_{k=1}^{m} a_{n+k} x^{k-1}.$$

Let P be a closed subset of P_0 and be such that if $A \in P$, $R(A, \cdot)$ is pole free, then there is $B \in P$ with $Q(B, \cdot) > 0$, $R(A, \cdot) = R(B, \cdot)$. Let there exist $A \in P$ with $\|f - F(A, \cdot)\| < \infty$. There exists an admissible best approximation with parameter in P to f.

Proof. Let $r \in R_{m-1}^{n-1}[a, b]$ have a pole. Let

$$L = \{x: |f(x) - \sigma(r(x))| \geqslant K\},$$

then

$$\|f - \sigma(r)\| \geqslant \int_{\sim L} \tau(f - \sigma(r)) + \int_L \alpha\, |f - \sigma(r)| \geqslant \alpha \left[\int_L |\sigma(r)| - \int_L |f|\right].$$

Let $N = \{x: |r(x)| \geqslant M\}$ then

$$\int_L |\sigma(r)| \geqslant \int_{L\cap(\sim M)} |\sigma(r)| + \beta \int_{L\cap M} |r|.$$

As the integral of $|r|$ over any neighbourhood of the pole is infinite, $\int_L |\sigma(r)| = \infty$ and $\|f - \sigma(r)\| = \infty$. It follows that if $\|f - F(A, \cdot)\| < \infty$, $R(A, \cdot)$ is pole-free, and there is admissible $R(B, \cdot)$ with $R(A, x) = R(B, x)$ for x not a zero of $Q(A, \cdot)$. Under Boehm's convention $R(A, \cdot) = R(B, \cdot)$.

The hypothesis on P on the theorem is satisfied by P_0 and $P_0 \cap P_2$. The example given previously for P_1 shows that the theorem does not hold for $P = P_1 \cap P_0$. The argument of the theorem cannot be extended to cover all transformers σ, for in the case $\sigma(x) = \log(x)$

$$\int_0^1 \log(1/x)dx = \int_\infty^1 \log(t)d(1/t) = \int_1^\infty (\log(t)/t^2)\, dt = [(1/t)(\log(t) - 1)]_1^\infty = 1$$

and approximations with a pole do not have infinite error.

Approximation on Finite Point Sets

The zero measure property does not hold if X has isolated points of positive measure and our previous theory does not apply. In the case X is a finite point set we can use an alternative convention to obtain existence. Let X be a p point set, say $1, 2, \ldots, p$ then the norm is of the form

$$\|g\| = \sum_{k=1}^{p} w_i \tau(g(i)), \qquad w_i > 0.$$

We define

$$\begin{aligned} F(A, i) &= \sigma(\infty), \qquad P(A, i) \neq 0, \qquad Q(A, i) = 0, \\ &= f(i), \qquad P(A, i) = Q(A, i) = 0, \end{aligned}$$

using a convention similar to that of Goldstein [4, pp. 84ff.]. The analog of Lemma 2 follows.

LEMMA 4. *If* $\| A^k \| \to \infty$ *and* $|\sigma(t)| \to \infty$ *as* $|t| \to \infty$ *then there exists an integer* i, $1 \leqslant i \leqslant p$ *such that*

$$|f(i) - F(A^k, i)| \to \infty.$$

Let $\hat{P}$ be the set of parameters A satisfying the normalization (1).

THEOREM. *Let* $\tau(t) \to \infty$ *as* $|t| \to \infty$ *and* 0 *be a minimum for* τ. *Let* $|\sigma(t)| \to \infty$ *as* $|t| \to \infty$. *Let* P *be a nonempty closed subset of* P_0 *or* $\hat{P}$. *There exists a best approximation to each bounded function on finite* X.

Proof. Let $\{A^k\} \subset P$ and $\{\|f - F(A^k, \cdot)\|\}$ be a decreasing sequence with $\lim \rho(f) = \inf\{\|f - F(A, \cdot)\| : A \in P\}$. From Lemma 4 it is seen that $\{\| A^k \|\}$ is a bounded sequence with accumulation point $A \in P$. By taking a subsequence if necessary we can assume that $\{A^k\} \to A$. If $Q(A, \cdot)$ vanishes on an integer i where $P(A, \cdot)$ does not, $\{P(A^k, i)/Q(A^k, i)\} \to \infty$ as $k \to \infty$, hence $\{w_i \tau(f(i) - F(A^k, i))\} \to \infty$, $\{\|f - F(A^k, \cdot)\|\} \to \infty$, contrary to hypothesis. Hence if $Q(A, i) = 0$, $P(A, i) = 0$ also and $F(A, i) = f(i)$. We have

$$\begin{aligned} w_i \tau(f(i) - F(A, i)) &= w_i \min \tau \leqslant w_i \tau(f_i - F(A^k, i)), && Q(A, i) = 0 \\ &= \lim_{k \to \infty} w_i \tau(f(i) - F(A^k, i)), && Q(A, i) \neq 0 \end{aligned}$$

Combining these we get

$$\|f - F(A, \cdot)\| \leqslant \lim_{k \to \infty} \|f - F(A^k, \cdot)\| = \rho(f).$$

OTHER TRANSFORMERS

There are transformers σ of interest which do not satisfy the condition $|\sigma(t)| \to \infty$ as $|t| \to \infty$. One such transformer is $\sigma(t) = \exp(t)$.

THEOREM. *Let* P, Q *have the zero measure property and there exist* B *with* $Q(B, \cdot) > 0$. *Let* $\tau(t) \to \infty$ *as* $|t| \to \infty$. *Let* $\sigma(t) \to \infty$ *as* $t \to \infty$ *and* $\sigma(t)$ *tend to a finite limit* Ω *as* $t \to -\infty$. *Let* P *be a nonempty closed subset of* P_0. *If we add* Ω *to the family of approximations, a best approximation exists to each bounded measurable function.*

Proof. Let $\|f - F(A^k, \cdot)\|$ be a decreasing sequence with limit $\rho(f)$. We have two possibilities. First, $\{\| A^k \|\}$ can be an unbounded sequence, then by taking a subsequence if necessary we can assume that $\|A^k\| \to \infty$. Define $B^k = A^k/\|A^k\|$ then $\|B^k\| = 1$ and $\{B^k\}$ has an accumulation point B, $\|B\| = 1$. Assume without loss of generality that $\{B^k\} \to B$. The sequence $\{(a_{n+1}^k, ..., a_{n+m}^k)\}$ is bounded and has an accumulation point C, assume that the sequence converges to C. By the normalization (1), $Q(C, \cdot) \neq 0$. We claim that for x not a zero of $Q(C, \cdot)$, $P(B, x)/Q(C, x) \leq 0$. Suppose not, let $P(B, x)/Q(C, x) > 0$ then there is $\epsilon > 0$ and a neighborhood N of x such that $P(B, y)/Q(C, y) > \epsilon$ for $y \in N$, hence for all k sufficiently large $R(A^k, y) > \|A^k\| \epsilon/2$ for $y \in N$. It follows that

$$\inf\{|f(y) - \sigma(R(A^k, y))| : y \in N\} \to \infty,$$

hence $\|f - R(A^k, \cdot)\| \to \infty$, giving a contradiction. Hence $P(B, \cdot)/Q(C, \cdot)$ is negative almost everywhere and $\sigma(R(A^k, \cdot)) \to \Omega$ almost everywhere. By Fatou's theorem [3, p. 59], $\|f - \Omega\| = \rho(f)$. The second possibility is that $\{\|A^k\|\}$ is bounded and that is handled by an earlier theorem.

In cases of practical interest Ω may never be best. Let us suppose that the range of σ is (Ω, ∞) and the family of rationals includes all constant functions. Then we would expect the range of f to be in (Ω, ∞) and then there exists a constant μ between Ω and f. If τ is strictly monotonic on $(-\infty, 0)$ and $(0, \infty)$, μ is a better approximation.

It appears that we may be able to guarantee the existence of a best admissible approximation only in the case of transformed linear approximation ($m = 1$). Consider for example the case where $X = [0, 1]$, $\sigma(t) = \exp(t)$, and R is a polynomial rational approximating function. The approximation $F(A, x) = \exp(-1/x)$ is continuous on $[0, 1]$, is the uniform limit of a sequence of admissible approximations, and corresponds to no admissible approximation.

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